Array shading for a broadband constant directivity transducer

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The theory of a broadband constant directivity transducer is developed. The transducer is an array of isophase, omnidirectional elements on an acoustically transparent spherical surface. It is shown that, with appropriate amplitude shading of the array elements, the beam pattern has no side lobes and the directivity is constant at all frequencies above a cutoff frequency (determined by the beam width and the radius of the sphere.) A shading function is derived, which consists of a simple linear combination of powers of cos θ, and several beam patterns are calculated.

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INTRODUCTION

A constant beamwidth transducer, that is, a transducer whose beam pattern is independent of frequency over a wide frequency range, is desirable for many applications in ultrasonics and underwater acoustics. Some examples of possible applications for such a transducer are (i) broadband echo ranging, (ii) high data rate communication, and (iii) nondestructive ultrasonic testing, medical diagnosis, and materials research. Constant beamwidth transducers designed so far involve the use of arrays of elements either interconnected by elaborate compensating networks[1] or delay lines,[2] or arranged in a complicated three-dimensional pattern.[3] All these transducers exhibit constant beamwidth only over a limited bandwidth. In this paper, we show that it is possible to obtain a constant beamwidth transducer by suitable amplitude shading of an array of isophase, omnidirectional elements on an acoustically transparent spherical surface. This transducer has the advantage that its beam pattern has no side lobes, and is independent of frequency at all frequencies above some lower-bound value determined by the beamwidth and the radius of the sphere.

Amplitude shading of a spherical array as the basis of a constant beamwidth transducer was first suggested by W. J. Trott.[4] In Ref. 4 the transmitting current response[4(a)] and the free-field voltage sensitivity of shaded spherical arrays are also considered, while in this paper the discussion is limited to beam patterns.

In Sec. I, the conditions are derived for a constant beamwidth farfield radiation pattern from an array on an acoustically transparent sphere. These conditions are used to develop an optimum shading function. Some calculated beam patterns are presented in Sec. II. The results are compared in Sec. III with the recently proposed constant beamwidth transducer consisting of a shaded array on an acoustically rigid sphere.

I. THEORY

Consider a continuous distribution of sources on an acoustically transparent sphere. Then, each area element dA of the sphere is a monopole source of strength $S(\theta_0)\,dA$, where $S(\theta_0)$ is the source strength per unit area. The sources are amplitude shaded, so S is a function of the polar angle θ_0 of the area element. The acoustic pressure (from element dA) at some point $\mathbf{r}$ outside the sphere is[5]

$$dp = -ikc\rho\,(e^{ik|\mathbf{r}-\mathbf{r}_0|}/|\mathbf{r}-\mathbf{r}_0|)\,S(\theta_0)\,dA\,, \qquad (1)$$

where $\mathbf{r}_0$ is the position vector of the area element dA, c and ρ are the sound speed and density, respectively, of the medium in which the array is immersed, and k is the wavenumber. All sources are assumed to radiate in phase at the same angular frequency ω, and the $e^{-i\omega t}$ time factor is omitted from all expressions. It is convenient to work in spherical coordinates and accordingly, the Green's function in Eq. (1) is rewritten in terms of the spherical coordinates of $\mathbf{r}$ and $\mathbf{r}_0$. The total pressure at point $\mathbf{r}$ is (see, for example, Morse and Ingard[5])

$$p(r,\theta) = \rho c k^2 a \sum_{m=0}^{\infty} A_m j_m(ka)\,h_m(kr)\,P_m(\cos\theta), \qquad (2)$$

where a is the radius of the sphere, $P_m(\cos\theta)$, is a Legendre polynomial, and j_m, h_m are spherical Bessel and Hankel functions, respectively. It is convenient to take the beam axis as the reference direction for the polar angles θ and θ_0. Also, the shading function is independent of the azimuthal angle φ. Therefore, the coefficients A_m in the above series are independent of θ and φ, and are determined from the shading function as follows:

$$A_m = (m+\tfrac{1}{2})\int_0^{\pi} S(\theta_0)\,P_m(\cos\theta_0)\sin\theta_0\,d\theta_0. \qquad (3)$$

The expression for the farfield pressure is obtained by taking the limit of Eq. (2) as $r\to\infty$,

$$p_f(r,\theta) = \rho c k a \frac{e^{ikr}}{r}\{[-A_1 j_1(ka)P_1(\cos\theta) + A_3 j_3(ka)P_3(\cos\theta) - \cdots] + i[-A_0 j_0(ka)P_0(\cos\theta) + A_2 j_2(ka)P_2(\cos\theta) - \cdots]\}. \qquad (4)$$

For a constant beamwidth transducer we want the farfield pressure amplitude $|p_f|$ to be independent of ka over as wide a frequency range as possible. Consider what happens at ka high enough so that one can use the asymptotic form of $j_m(ka)$. Then,

$$p_f(r,\theta)\to \rho c\, e^{ikr}/r\{[A_1P_1(\cos\theta)+A_3P_3(\cos\theta)+\cdots]\cos(ka)$$
$$-i[A_0P_0(\cos\theta)+A_2P_2(\cos\theta)+\cdots]\sin(ka)\}. \quad (5)$$

The shading function $S(\theta)$ can also be expanded as a series of Legendre polynomials. It is convenient to express $S(\theta)$ as the sum of an even part $S_e(\theta)$ (even with respect to the variable $\cos\theta$) and an odd part $S_0(\theta)$ where,

$$S_e(\theta)=A_0P_0(\cos\theta)+A_2P_2(\cos\theta)+\cdots,$$

and

$$S_0(\theta)=A_1P_1(\cos\theta)+A_3P_3(\cos\theta)+\cdots. \quad (6)$$

From Eqs. (5) and (6) it follows that the farfield pressure amplitude can be expressed as

$$|p_f(r,\theta)|=(\rho c/r)[\{S_0(\theta)\cos(ka)\}^2+\{S_e(\theta)\sin(ka)\}^2]^{1/2}. \quad (7)$$

Finally, suppose the shading function is chosen so that

$$|S_0(\theta)|=|S_e(\theta)|,$$

Then,

$$|p_f(r,\theta)|=(\rho c/r)|S_0(\theta)| \quad (8)$$

and is *independent of ka*.

Obviously it is important to know at what values of ka one can approximate a spherical Bessel function of order m by its asymptotic form. It can be shown[6] that the asymptotic form applies when $(ka)^2 \gg m^2-\frac{1}{4}$. Thus, the higher the order n the higher the value of ka before $j_m(ka)$ approaches its asymptotic value. From this fact, and from the results presented in the previous paragraph, emerge the following two criteria for amplitude shading on an acoustically transparent sphere (to achieve constant beamwidth).

(i) Choose a shading function whose expansion, in Legendre polynomials, involves the least number of terms possible for the given beamwidth. Alternately, if m_u is the highest-order term in Eq. (4) which makes an observable contribution to $p_f(r,\theta)$, choose $S(\theta)$, such that m_u has the lowest possible value.

(ii) Choose $S(\theta)$ such that its odd and even parts are equal in magnitude. Note that this criterion is *automatically satisfied*, if the shading function is finite in the upper hemisphere ($0\le\theta\le\pi/2$) and zero in the lower hemisphere ($\pi/2\le\theta\le\pi$). The only way to obtain $S(\theta)=0$ in the range $\pi/2\le\theta\le\pi$ is for $S_o(\theta)$, $S_e(\theta)$ to be equal in amplitude but have opposite sign.

Note that when criteria (i) and (ii) are satisfied, it follows from Eq. (8) that the beam pattern will be the same as the shading function. Therefore, to eliminate side lobes it is necessary to choose an $S(\theta)$ which decreases smoothly to zero as a function of θ.

According to Eq. (8) the beam pattern will be symmetrical about the $\theta=90°$ plane, with equal farfield pressure amplitude in the forward ($\theta=0°$) and back ($\theta=180°$) directions.

The second of the above criteria was the one mainly used as a guide in developing an acceptable shading function. Initially attempts were made to obtain a suitable shading function by direct linear combination of a limited number of Legendre polynomials. However, the individual $P_m(\cos\theta)$ are strongly oscillatory functions and it is not immediately obvious how to combine them to cancel in the lower hemisphere. A more convenient starting function is $\cos^n\theta$, which varies smoothly as a function of θ and, as shown below, simple linear combinations of powers of $\cos\theta$ can be developed which, to a very good approximation, satisfy criterion (ii). The simplest combination of powers of $\cos\theta$ which tends to zero in the lower hemisphere is

$$f_n(\theta)=\tfrac{1}{2}(1+\cos\theta)\cos^n\theta. \quad (9)$$

In the lower hemisphere, f_n has either a shallow maximum or a minimum depending on whether n is even or odd. The magnitude of this peak is small. For example, when $n=1$, the peak magnitude of $f_1(\theta)$, in the range $\pi/2\le\theta\le\pi$, is 18 dB below the value of f_1 in the forward direction ($\theta=0°$); and as n increases, the cancellation between the two terms in $f_n(\theta)$ becomes even stronger. Further cancellation is achieved by forming a linear combination of $f_n(\theta)$ and $f_{n+1}(\theta)$ and choosing the coefficients, so that the peak value of $f_n(\theta)$ is exactly canceled by $f_{n+1}(\theta)$. Let θ' be the value of θ at which $f_n(\theta)$ has a maximum (or minimum) in the lower hemisphere. From Eq. (9) it follows that,

$$\cos\theta'=-[n/(n+1)].$$

Let $r=|f_{n+1}(\theta')|/|f_n(\theta')|$ be the ratio of amplitudes of f_{n+1} and f_n at θ'. Then the appropriate linear combination of f_n and f_{n+1}, normalized to unity at $\theta=0°$, is

$$S_n(\theta)=\frac{1}{1+r}\,[rf_n(\theta)+f_{n+1}(\theta)]$$
$$=\frac{n}{2(2n+1)}\cos^n\theta+\tfrac{1}{2}\cos^{n+1}\theta+\frac{n+1}{2(2n+1)}\cos^{n+2}\theta. \quad (10)$$

This function is close to zero over the entire range $\pi/2\le\theta\le\pi$. When $n=1$, the peak magnitude of $S_n(\theta)$ in the lower hemisphere is 36 dB below unity, and decrease further with increasing n.

The series expansion of $\cos^n\theta$ in Legendre polynomials[6] involves only polynomials of order less than or equal to n. Thus, the highest order term in the series expansion of $S_n(\theta)$ is of order $n+2$. It is possible that, for a given beamwidth, other shading functions can be developed which involve an even smaller number of Legendre polynomials than the corresponding $S_n(\theta)$. What can be said at this stage is that the calculated beam patterns, for $S_n(\theta)$ shading, show a constant beamwidth and absence of side lobes. These beam patterns are shown in the next section.

II. CALCULATED BEAM PATTERNS

For several shading functions $S_n(\theta)$ beam patterns were calculated by numerically evaluating and summing the terms in Eq. (4). A computer program, developed by Van Buren,[7] was used. In this program the coefficients A_m are obtained by numerical integration (Gaussian quadrature) of the integral in Eq. (3), and the Legendre polynomials and Bessel functions are obtained by use of appropriate recursion relations.

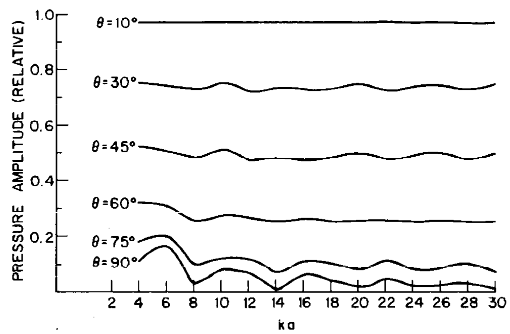

FIG. 1. Calculated relative farfield pressure amplitude, as a function of ka, for various values of the polar angle θ. The shading function is $S_1(\theta)$.

One shading function used was $S_1(\theta)$. From Eq. (10) it follows that

$$S_1(\theta)=\frac{1}{6}\cos\theta+\frac{1}{2}\cos^2\theta+\frac{1}{3}\cos^3\theta\,. \tag{11}$$

Figure 1 shows the calculated farfield pressure, as a function of ka, for various values of the polar angle θ. It can be seen that, for $ka \gtrsim 8$, the beamwidth becomes approximately independent of frequency. Such pressure amplitude variations as remain are small, less than 0.5 dB, in the range $0° \leq \theta \leq 60°$. Larger amplitude variations are observed at $\theta \gtrsim 75°$, but here the total signal is small (at least 20 dB below the signal in the forward direction.)

The beam pattern obtained with $S_1(\theta)$ shading is the broadest possible constant beamwidth pattern for this type (powers of $\cos\theta$) shading function. Examination of the data in Fig. 1 shows that for $S_1(\theta)$ shading the beam pattern is approximately $\cos^2\theta$. For $S_n(\theta)$ with higher values of n, the beam pattern becomes progressively narrower. From the results in Sec. 1, it follows that the beam pattern for $S_n(\theta)$ is $\cos^{n+1}\theta$. Therefore, the -6 dB total beamwidth, $2\theta_{1/2}$, for a given $S_n(\theta)$, can be determined from the equation,

$$\cos^{n+1}\theta_{1/2}=0.5. \tag{12}$$

The numerical solution of Eq. (12) is plotted in Fig. 2.

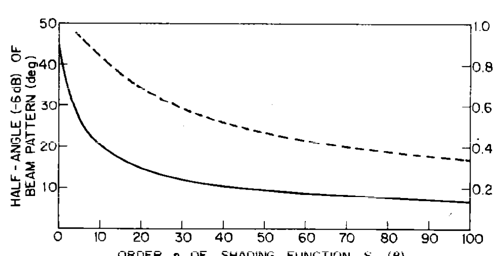

FIG. 2. The full line shows how the half-angle (-6 dB point) of the beam pattern depends on the order n of the shading function $S_n(\theta)$. The dashed line shows the ratio c/a as a function of n, where c is the radius of the shaded spherical cap and a is the radius of the sphere.

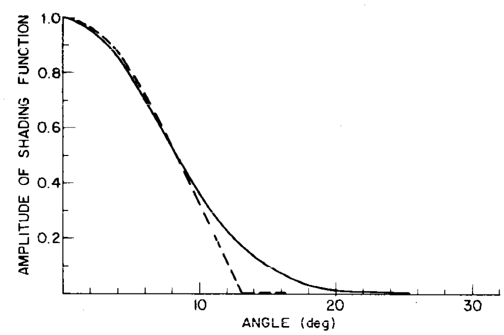

FIG. 3. Comparison of the shading functions $S_{65}(\theta)$ (full line) and $P_{10}(\cos\theta)$ (dashed line).

Calculations were also made for the very narrow shading function.

$$S_{65}(\theta)=\frac{65}{262}\cos^{65}\theta+\frac{1}{2}\cos^{66}\theta+\frac{66}{262}\cos^{67}\theta. \tag{13}$$

The value $n=65$ was chosen because $S_{65}(\theta)$ approximates closely the first peak of the Legendre polynomial $P_{10}(\cos\theta)$, at least down to the -6 dB point. This can be seen in Fig. 3, where the two functions are compared. [The reasons for comparison with $P_{10}(\cos\theta)$ shading are given in the next section.] The calculated farfield pressure amplitude is plotted in Fig. 4. It can be seen that, for $ka \gtrsim 180$, the beam pattern becomes essentially independent of frequency. Also there are no side lobes.

In theory, the shading function $S_n(\theta)$ extends over the entire sphere. However, calculations show that for large n the shading function can be terminated at the angle for which $S_n(\theta)\sim\frac{2}{1000}$, and such termination does not affect the beam pattern. Therefore, in practice, for narrow beams it is sufficient to shade a spherical cap of radius c, smaller than the radius a of the sphere. The ratio c/a is plotted, as a function of n, in Fig. 2.

Finally, it is of interest to determine at what value of ka [for a given $S_n(\theta)$] the farfield pressure attains its asymptotic, constant beamwidth, value. For $S_n(\theta)$ shading, the highest term in the series expansion [Eq. (4)] for the farfield pressure is of order $n+2$. As stated in

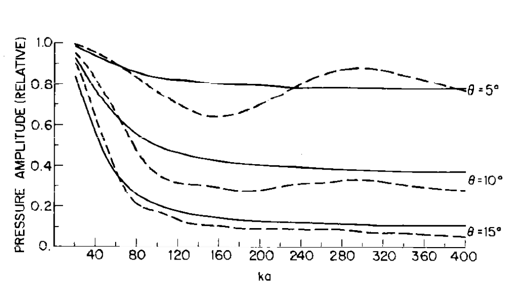

FIG. 4. Calculated relative farfield pressure amplitude, as a function of ka, for various values of the polar angle θ. The full and dashed lines are for $S_{65}(\theta)$ and $P_{10}(\cos\theta)$ shading functions, respectively.

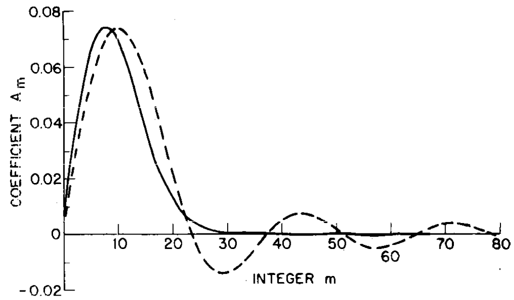

FIG. 5. Values of the coefficient A_m in the Legendre series expansion of the shading function. The full and dashed lines are for $S_{65}(\theta)$ and $P_{10}(\cos\theta)$ shading functions, respectively.

Sec. I, the spherical Bessel function $j_{n+2}(ka)$ attains its asymptotic form when $(ka)^2 \gg (n+2)^2 - \frac{1}{4}$. Our calculations show that the farfield pressure attains its asymptotic value when $ka \sim 3(n+2)$.

III. DISCUSSION

Recently, Rogers and Van Buren[8] have presented the theory of a broadband constant beamwidth transducer achieved by amplitude shading the velocity distribution on an acoustically rigid sphere. It is of interest to compare their results with the data presented in this paper. When the radial velocity distribution on the rigid sphere is $v(\theta)\, e^{-i\omega t}$, then the radiated farfield pressure is[5]

$$p_f(r,\theta) = \rho c\, e^{ikr}/kr \sum_{m=0}^{\infty} A_m P_m(\cos\theta) \frac{1}{i^m h_m'(ka)}\,, \tag{14}$$

where h_m' is the derivative of the spherical Hankel function, and the time dependence $e^{-i\omega t}$ has been omitted. The A_m are the coefficients in the Legendre expansions of the radial velocity amplitude shading function and are obtained from Eq. (3) by substituting $v(\theta)$ for $S(\theta)$. At values of ka large enough so that the asymptotic form of $h_m'(ka)$ can be substituted in Eq. (14), the farfield pressure is

$$p_f(r,\theta) = \rho c a\,(e^{ik(r-a)}/r)\, v(\theta), \tag{15}$$

and the beam pattern is independent of frequency.

Comparison of the conditions leading to Eq. (15) with those for Eq. (8) shows that only criterion (i) of Sec. I applies for amplitude shading of an array on an acoustically rigid sphere (to achieve constant beamwidth). However, as explained in Sec. I, if the array is confined to the upper hemisphere $v(\theta)=0$ in the range $\pi/2 \le \theta \le \pi$, then criterion (ii) will be automatically satisfied. Therefore, for narrow beamwidths the same conditions apply to amplitude shading of arrays on acoustically rigid and acoustically transparent spheres, and the shading function, which is usable for one array, should also apply to the second case.

The constant beamwidth transducer proposed by Rogers and Van Buren is a spherical cap of half angle α_n shaded so that the normal velocity is given by $v(\theta) = P_n(\cos\theta)$, where α_n is the first zero of the Legendre polynomial P_n. The array is terminated at α_n, so that $v(\theta)=0$ for $\theta > \alpha_n$. In Fig. 4 the beam patterns for $P_{10}(\cos\theta)$ and $S_{65}(\theta)$ shading functions are compared, where $S_{65}(\theta)$ was chosen because it approximates closely the first peak of $P_{10}(\cos\theta)$ down to the -6 dB point. It can be seen from Fig. 4 that the beam pattern for $S_{65}(\theta)$ is a closer approximation to a constant beamwidth pattern. For example, at $\theta = 5°$ the farfield pressure for $P_{10}(\cos\theta)$ shading shows a peak to peak variation of 2.7 dB in the range $80 \le ka \le 400$. Further comparison is made in Fig. 5, where the coefficients A_m in the series expansion in Legendre polynomials of the two shading functions are plotted as a function of m. It can be seen that the series for $S_{65}(\theta)$ involves fewer coefficients, and therefore this shading function satisfies criterion (i) more closely than does $P_{10}(\cos\theta)$. The improvement in constant beamwidth characteristics is at the expense of transducer size. The ratio c/a, where c is the radius of the shaded spherical cap, is 0.42 and 0.23 for $S_{65}(\theta)$ and $P_{10}(\cos\theta)$ shading, respectively.

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[4](a) *Note added in proof*: The transmitting current response data was extracted from an early draft of Ref. 8.
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